Matrix division is the opposite of multiplication.

→ Many problems can be written as matrix division.

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EX: Solve the system $\left\{\begin{array}{l} 2x + 4y = 8 \\ -3x + 4y = -2 \end{array}\right\}$

This is equivalent to $\begin{bmatrix}2&4\\-3&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-2\end{bmatrix}$

so $\begin{bmatrix}x\\y\end{bmatrix}=\dfrac{\begin{bmatrix}8\\-2\end{bmatrix}}{\begin{bmatrix}2&4\\-3&4\end{bmatrix}}$ Matrix Division!

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Note: We must be careful about which side we are dividing on.

$$\begin{bmatrix}2&4\\-3&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-2\end{bmatrix} \xrightarrow{\text{Divide on left}} \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2&4\\-3&4\end{bmatrix}\backslash\begin{bmatrix}8\\-2\end{bmatrix}$$

(usually we want to divide on left)

$$\begin{bmatrix}x&y\end{bmatrix}\begin{bmatrix}2&4\\-3&4\end{bmatrix}=\begin{bmatrix}-4&12\end{bmatrix} \xrightarrow{\text{Divide on right}} \begin{bmatrix}x&y\end{bmatrix}=\begin{bmatrix}-4&12\end{bmatrix}\big/\begin{bmatrix}2&4\\-3&4\end{bmatrix}$$

Just as with multiplication, we compute division by converting to a system of equations.

(This is usually a very bad method...)

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EX: Divide $\begin{bmatrix}2&4\\-3&4\end{bmatrix}\backslash\begin{bmatrix}8\\-2\end{bmatrix}$

It is equivalent to solve

$$\begin{bmatrix}2&4\\-3&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}8\\-2\end{bmatrix} \rightsquigarrow \left\{\begin{array}{l} 2x + 4y = 8 \\ -3x + 4y = -2 \end{array}\right\}$$

(As children, we learned to solve this by adding equations to eliminate a variable and then substituting....)

$$\begin{array}{rl} & 2x + 4y = 8 \\ -( & -3x + 4y = -2) \\ \hline & 5x \quad\quad = 10 \end{array} \Rightarrow \boxed{x=2}$$

subst. $2(2) + 4y = 8 \Rightarrow y = 1$

$\Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

So $\begin{bmatrix}2&4\\-3&4\end{bmatrix}\backslash\begin{bmatrix}8\\-2\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$

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Check: $\begin{bmatrix}2&4\\-3&4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}4+4\\-6+4\end{bmatrix}\overset{\checkmark}{=}\begin{bmatrix}8\\-2\end{bmatrix}$

Problem: Messing about with equations quickly becomes difficult when you have more than two variables and/or more than two equations.

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Solution #1: Instead of converting to a system of equations and adding multiples of equations together, convert to an augmented matrix and use "Gaussian Elimination"

divide on left

$$\begin{bmatrix} 2 & 4 \\ -3 & 4 \end{bmatrix} \backslash \begin{bmatrix} 8 \\ -2 \end{bmatrix} \longleftrightarrow \left[\begin{array}{cc|c} 2 & 4 & 8 \\ -3 & 4 & -2 \end{array}\right] \mapsto \ldots \text{ etc.}$$

(You might have learned this as a child.)

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We will instead learn to divide using a method which is more powerful and more interesting.

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Solution #2: Factor the matrix into its "LU-Decomposition" and use this to compute matrix division.

The idea for the LU-Decomposition comes from the following

FACT: Some matrices are easier to divide by than others.

→ Instead of doing one hard division, we will do two easy divisions!

Def: The main diagonal of a matrix starts at the upper-left corner and moves diagonally

EX: $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}$ Main Diagonal

EX: $\begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 0 \end{bmatrix}$ Main Diagonal

EX: $\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 3 & -2 \end{bmatrix}$ Main Diagonal

Def: An upper-triangular ("upper Δ") matrix is 0 below the main diagonal.

A lower-triangular ("lower Δ") matrix is 0 above the main diagonal.

EX $\begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ Upper-Triangular

EX: $\begin{bmatrix} 2 & 3 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$ Upper-Triangular

EX: $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ Lower-Triangular

EX: $\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 3 & 0 \end{bmatrix}$ Lower-Triangular

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Dividing by triangular matrices can be done using (forward or back)-substitution (except for one "special case" to be discussed later)

EX: Divide $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \backslash \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

To solve this we will convert to equations:

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} \rightsquigarrow \left\{ \begin{aligned} 2x \qquad\qquad &= 6 \\ 3x + 4y \qquad &= 5 \\ -x + 2y + 3z &= 1 \end{aligned} \right\}$$

First equation determines x.

→ Plug x value into next equation to get y.

→ Plug x & y values into next equation to get z.

$$\left\{ \begin{aligned} 2x \qquad\qquad &= 6 \rightsquigarrow x = 3 \\ 3x + 4y \qquad &= 5 \rightsquigarrow 3 \cdot 3 + 4y = 5 \\ & \qquad\qquad\quad y = -1 \\ -x + 2y + 3z &= 1 \rightsquigarrow -3 + 2(-1) + 3z = 1 \\ & \qquad\qquad\quad z = 2 \end{aligned} \right.$$

So $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \backslash \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

(This is called "(forward) substitution")

Check: $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 9-4 \\ -3-2+6 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$

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Actually what is needed is not △ shape but that the last <u>nonzero</u> element in lower rows is further to the right.

→ The matrix should have a "<u>stair-step</u>" shape

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 3 & 0 \end{bmatrix}$$

"stair-step" shape

EX: Divide $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \diagdown \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix}$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix} \rightsquigarrow \left\{ \begin{aligned} 2x + 3y - z &= 2 \\ 4y + 2z &= -2 \\ 3z &= -9 \end{aligned} \right\}$$

(Start with the last equation & last variable — substitute upwards ("back-substitution"))

$$3z = -9 \rightsquigarrow z = -3$$
$$4y + 2z = -2 \rightsquigarrow 4y + 2(-3) = -2$$
$$y = 1$$
$$2x + 3y - z = 2 \rightsquigarrow 2x + 3(1) - (-3) = 2$$
$$x = -2$$

So $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix} \diagdown \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$

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Check: $\begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4+3+3 \\ 4-6 \\ -9 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 2 \\ -2 \\ -9 \end{bmatrix}$

## LU-Decomposition.

All matrices can be factored as a product

$$A = L \cdot U$$

where $U$ is an upper-triangular matrix with the same size as $A$

$L$ is a lower-triangular matrix which is square (#rows(L) = #cols(L) = #rows(A)) with all 1 on the diagonal

→ Knowing L & U is like knowing $\frac{1}{A}$...

(Computing L & U is not difficult. But first I will give an example showing how to use $A = L \cdot U$ to divide.)

EX: Divide $\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix} \diagdown \begin{bmatrix} -1 \\ -7 \\ 0 \end{bmatrix}$

if $\underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 4 & -3 & 8 \\ -2 & 0 & -3 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 2 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$

(Instead of dividing by A, divide by L & U)

$$\begin{bmatrix}2&-1&3\\4&-3&8\\-2&0&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\-7\\0\end{bmatrix}$$

Convert from A to LU decomp.

$$\underbrace{\begin{bmatrix}1&0&0\\2&1&0\\-1&1&1\end{bmatrix}}\begin{bmatrix}2&-1&3\\0&-1&2\\0&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\-7\\0\end{bmatrix}$$

Divide by left-most matrix (L) first.

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&1&1\end{bmatrix}\Big\backslash\begin{bmatrix}-1\\-7\\0\end{bmatrix}$$

Convert to multiplication to get the answer.

Note: Do not use $\begin{bmatrix}x\\y\\z\end{bmatrix}$ because that is final answer

$$\Downarrow$$

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-1\\-7\\0\end{bmatrix} \rightsquigarrow \begin{aligned}a&=-1\\b&=-5\\c&=4\end{aligned}$$

Now divide by right-most matrix (U).

$$\begin{bmatrix}2&-1&3\\0&-1&2\\0&0&-2\end{bmatrix}\Big\backslash\begin{bmatrix}-1\\-5\\4\end{bmatrix}$$

Convert to multiplication to get the answer.

$$\Downarrow$$

$$\begin{bmatrix}2&-1&3\\0&-1&2\\0&0&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\-5\\4\end{bmatrix} \rightsquigarrow$$

(back-subst.)

$$z=-2$$
$$y=1$$
$$x=3$$

Answer: $\begin{bmatrix}3\\1\\-2\end{bmatrix}$

Check: $\begin{bmatrix}2&-1&3\\4&-3&8\\-2&0&-3\end{bmatrix}\begin{bmatrix}3\\1\\-2\end{bmatrix}=\begin{bmatrix}6-1-6\\12-3-16\\-6+6\end{bmatrix}\overset{\checkmark}{=}\begin{bmatrix}-1\\-7\\0\end{bmatrix}$

EX: Divide

$$\begin{bmatrix}3&1&-2&2\\9&1&-5&6\\3&-3&0&5\end{bmatrix}\Big\backslash\begin{bmatrix}3\\7\\-7\end{bmatrix}$$

if $\underbrace{\begin{bmatrix}3&1&-2&2\\9&1&-5&6\\3&-3&0&5\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0\\3&1&0\\1&2&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}3&1&-2&2\\0&-2&1&0\\0&0&0&3\end{bmatrix}}_{U}$

Solve $\underbrace{\begin{bmatrix}1&0&0\\3&1&0\\1&2&1\end{bmatrix}}\underbrace{\begin{bmatrix}3&1&-2&2\\0&-2&1&0\\0&0&0&3\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix}}=\begin{bmatrix}3\\7\\-7\end{bmatrix}$

Divide by L

$$\begin{bmatrix}1&0&0\\3&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}3\\7\\-7\end{bmatrix}\rightsquigarrow\begin{aligned}a&=3\\b&=-2\\c&=-6\end{aligned}$$

Divide by U

$$\begin{bmatrix}3&1&-2&2\\0&-2&1&0\\0&0&0&3\end{bmatrix}\begin{bmatrix}w\\x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\-2\\-6\end{bmatrix}$$

Back substitution: $3z=-6 \rightsquigarrow z=-2$

$$-2x+y=-2 \rightsquigarrow y=\text{free}$$
$$x=1+\tfrac{1}{2}y$$

$$3w+x-2y+2z=3$$
$$3w+(1+\tfrac{1}{2}y)-2y+2(-2)=3$$
$$\rightarrow w=2+\tfrac{1}{2}y$$

Answer: $\begin{bmatrix}2+\frac{1}{2}y\\1+\frac{1}{2}y\\y\\-2\end{bmatrix}=\begin{bmatrix}2\\1\\0\\-2\end{bmatrix}+\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\1\\0\end{bmatrix}y$